A Fuzzy System Approach for Choosing Public Goods Game Strategies

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Abstract—Replicator equations are regularly used to predict how strategies evolve in social dilemmas. These predictions are based on comparisons between the fitness of a strategy and the average population fitness. Unfortunately, fitness comparisons alone don’t provide much insight into how or why individuals choose to cooperate. To overcome this limitation in replicator equations we developed a zero order Seguno fuzzy system to model individual player decisions in a public goods game. Our simulation results qualitatively match those predicted by the replicator dynamics which validates the approach. This new methodology provides a framework for studying individual decision making in social dilemmas.

I. INTRODUCTION

Why humans cooperate, particularly with unrelated individuals, is not fully understood. Natural selection favors defectors who free ride by exploiting the cooperators. A number of theories such as direct and indirect reciprocity or kin selection have been posited to explain persistent cooperation among unrelated individuals. Unfortunately, these theories fail when the game is not repeated and individual reputations are ignored.

The public goods game (PGG) is a widely studied $N$-player game used to evaluate how strategies persist in social dilemmas where individuals must tradeoff personal gain versus group benefits. Cooperators ($C$) contribute to the public good which benefits the group whereas defectors ($D$) contribute nothing but still share the public good. Thus, defectors free ride on the contributions of others. Studies show defectors always ultimately prevail in populations consisting solely of cooperators and defectors.

Researchers have begun looking at alternative strategies (other than just ‘cooperate’ or ‘defect’) to see how cooperation levels evolve in populations of size $N > 2$. Altruistic punishment has emerged as one of those strategies. An altruistic punisher acts like a cooperator but also pays a personal cost to impose a punishment—e.g., a fine—on defectors. The punisher expects nothing in return including future reciprocity. The idea is a sufficiently punished defector will switch to some other strategy. If some individuals become altruistic punishers ($P$), and they are common, then the group benefits because defectors eventually stop defecting. Another recently investigated alternative strategy is the “loner” ($L$) who does not participate in the PGG but instead gets a lower, albeit guaranteed return. Loners introduce oscillations in cooperator and defector frequencies because the public good payoff is no longer attractive whenever there are few cooperators [1].

Replicator equations are widely used to study how strategy frequencies change over time; they are considered fundamental in explaining evolutionary dynamics [2]. The relative abundance (frequency) of a strategy $s_i$ is denoted by $x_i$. The fitness of $s_i$ ($F_i$) is compared to the average fitness in the population $\bar{F}$. The equations predict $x_i$ grows if $F_i > \bar{F}$, decreases if $F_i < \bar{F}$ and remains unchanged if $F_i = \bar{F}$. Yet despite their widespread predictive use, they have an inherent flaw: they provide limited insight into why or how individuals make strategy choices in social dilemmas.

To overcome this replicator equation limitation we have adopted an entirely different approach. In our work we have modeled individual player decisions as a zero-order Sugeno fuzzy system. Each player has a set of fuzzy rules tailored to his current strategy choice ($C$, $D$, $P$ or $L$). The consequent of these rules induces an emotional response to the strategy choices of other players in the previous round. A player changes to a different strategy if the emotional response is strong enough. The new strategy choice is made based on how a rational individual might respond. Our PGG results qualitatively match what replicator equations predict which validates this approach. The difference is with our approach the results can be reversed engineered (they are based on individual decisions) thereby providing a much improved framework for studying human cooperation in social dilemmas.

The paper is organized as follows. Section II provides background material on the PGG and replicator equations. The fuzzy system model is described in Section III. Simulation results are shown in Section IV and analyzed in Section V. Future work is discussed Section VI.

II. BACKGROUND

In this section the PGG is formally defined. A PGG can be defined in different ways. In this work we adopted the definition used by Fowler [3]. Also included is a description of replicator equations and their use in PGGs.

A. The PGG

Suppose a large population has an opportunity to create a public good. Cooperators ($C$) contribute an amount $c$ to increase the public good by an amount $b$ while defectors
(D) contribute nothing. The public good is then redistributed equally to everyone regardless of whether or not they contributed anything. Let \( x_c \) be the frequency of C individuals and \( x_L \) the frequency of D individuals in the population. Then the expected payoffs, which are a measure of fitness, are \( bx_c - c \) for cooperators and \( bx_L \) for defectors. Defectors are therefore free riders who exploit the contributions of others. The best individual outcome is to choose D regardless of what others do. However, the best group outcome occurs when everyone chooses C.

Many research studies have used this type of PGG to investigate why cooperation persists in human populations. The ultimate causes are still not well understood but two strategies have emerged that do tend to promote cooperation: loners or “loners” (L) and altruistic punishers (P).

Loner are busy doing other things. They contribute nothing to the public good nor do they share in its redistribution. Instead, they always receive a fixed albeit guaranteed payoff which is less that of all-C population. Nevertheless, this payoff is attractive whenever \( x_c \) is low. Adding loners with frequency \( x_L \) to the population requires modification of the payoffs given previously. Now the cooperators payoffs are \( bx_c / (1 - x_c) - c \); for defectors \( bx_c / (1 - x_c); \) and \( \sigma < b - c \) for loners. This results in a rock-paper-scissors type of evolution because the PGG payoffs are less attractive whenever \( x_c \) is low.

Finally, we add punishers as a fourth strategy. Punishers contribute \( c \) to the public good and receive a payoff but they also punish free riders. This punishment is intended to get defectors to start contributing. Punishers pay a cost \( \xi \) to inflict a punishment \( \beta \) on free riding defectors. But cooperators are also free riders because they benefit from the punishment defectors get but don’t pay any of the associated costs. Punishers pay an additional cost \( a\xi \) to levy a punishment \( a\beta \) on free riding cooperators where \( 0 < a < 1 \).

Defectors are called 1st order free riders while cooperators who do not punish are called 2nd order free riders. The penalty (and associated cost) is lower for a 2nd order free rider because failure to punish is not as bad as failure to contribute. Since loners do not participate in the PGG they are ignored by punishers. However, a further modification of the payoffs is now required because both C and P players are contributing to the public good and free rider penalties and costs are applied. The payoffs are now for cooperators \( b(x_c + x_P)/ (1-x_c) - c - a\alpha x_c \); for defectors \( b(x_c + x_P) / (1-x_c) - \beta x_L \); for punishers \( b(x_c + x_P) / (1-x_c) - c - a\xi x_c - \xi x_P \); and \( \sigma < b - c \) for loners.

### B. Replicator Equations

Players in a PGG update their strategies from time to time. A convenient way of depicting the evolution of strategies is with a simplex. Suppose a population can have \( M \) possible strategies where \( x_i \in [0,1] \) is the frequency of strategy \( i \) and \( \sum x_i = 1 \). The evolution of the M strategies in the population can be depicted as a trajectory in a \((M - 1)\)-simplex. This simplex is invariant under replicator dynamics because any trajectory exists only on the simplex. Every point in the simplex has coordinates \([x_1 \ x_2 \ldots x_M]\) so each point specifies a unique mixture of the \( M \) strategies. When the population is of infinite size the evolution is governed by the standard replicator equations which are 1st-order differential equations of the form

\[
\dot{x}_i = x_i \left( F_i - \bar{F} \right) \tag{1}
\]

\( F_i \) is the fitness of strategy \( i \) and \( \bar{F} \) is the mean population fitness. (For simplicity here we equate fitness with payoffs.) The frequency of strategy \( i \) increases if the term in parenthesis is positive; remains unchanged if zero; and decreases if negative. Infinite population trajectories are continuous and can pass through any point in the simplex.

All populations in nature are finite so a discrete version of the replicator equations is needed. Let \( k_i \) be the number of players choosing strategy \( i \) \( \in \{C, P, D, L\} \). In a finite population the frequency of strategy \( i \) at time \( t \) is \( x_{i}^t = k_i/N \) where \( N \) is the population size and \( \sum x_i = N \). The strategy evolution over time is now given by the discrete replicator equation

\[
x_{i}^{t+1} = x_{i}^t \left( \frac{\pi_{i}^t}{\bar{\pi}^t} \right) \tag{2}
\]

where \( \pi_{i}^t \) is the payoff to a player at time \( t \) who chose strategy \( i \) and \( \bar{\pi}^t \) is the average population payoff at time \( t \). Multiplying both sides of Eq. (2) by \( N \) gives an equivalent form

\[
k_{i}^{t+1} = k_{i}^t \left( \frac{\pi_{i}^t}{\bar{\pi}^t} \right) \tag{3}
\]

As in the continuous form the term in parenthesis determines frequency changes. The frequency increases if \( \pi_i > \bar{\pi} \); doesn’t change if \( \pi_i = \bar{\pi} \) and decreases if \( \pi_i < \bar{\pi} \). Quantization is necessary to ensure \( \sum x_i = N \). (See [4] for further details and a simple quantization algorithm.) Unlike infinite populations, with finite populations the simplex trajectories can only move between specific points—i.e., points where \( \sum x_i = N \) (see Figure 1). Consequently, trajectories are piecewise linear for the finite population case.
III. Model Description

There is a limitation in the discrete form of the equations that does not exist in the continuous form. In a PGG the payoff of a strategy (for other than \( L \) players) equals the amount returned minus any contribution, punishments or costs. That payoff could be negative. Negative payoffs pose no problem in continuous replicators but in discrete replicators it yields negative frequencies. To prevent a negative righthand side in Eq. (2) it is necessary to redefine the payoffs but in a way consistent with the continuous form of the replicator equations. The payoffs \( \pi_i^{(t)} \) in finite populations are now defined for each player type as follows:

\[
\pi_i^{(t)} = \begin{cases} 
\phi \cdot e^{-\lambda(c+\alpha bx_i)} & \text{cooperators} \\
\phi \cdot e^{-\lambda(c+\alpha bx_i+\xi x_v)} & \text{punishers} \\
\phi \cdot e^{-\lambda(\beta x_i)} & \text{defectors} \\
\sigma & \text{loners} 
\end{cases}
\]

where \( \phi = b \left( \frac{x_c + x_p}{1 - x_i} \right) \). \( \beta \) is the punishment administered for which punishers pay a cost \( \xi \). The amount returned to each \( C, P \) and \( D \) player is \( b \left[ \frac{x_c + x_p}{1 + x_i} \right] \) but the exponential term reduces it to account for any contribution, punishments or costs. Now \( \pi_i^{(t)} \geq 0 \). \( a \) and \( \lambda \) are scaling parameters with \( 0 < a < 1 \). In an all-\( C \) population—i.e., where \( x_c = 1 \)—the payoff is \( b - c \). Choosing \( \lambda = (1/c) \ln(1 - c/b) \) makes the redefined payoff equal to \( b - c \) in an all-\( C \) finite population. \( L \) players always get a fixed payoff \( \sigma < b - c \).

To summarize the costs and punishments,
1) punishers inflict a punishment \( e^{-\lambda c x_i} \) on a defector
2) punishers inflict a punishment \( e^{-\lambda a bx_i} \) on a cooperator
3) punishers pay a cost \( e^{-\lambda c x_v} \) for punishing defectors
4) punishers pay a cost \( e^{-\lambda a \xi x_c} \) for punishing cooperators

Every player has an emotional level and an emotional threshold. The emotional levels are initialized to zero while the thresholds are randomly initialized between 5 and 10. Players see their emotional levels increase as the game is played (described shortly) and change strategies when their emotional level exceeds their emotional threshold. The reason for different thresholds is people react differently to the same set of events. For example, a situation that causes outrage in one individual may only annoy another.

A zero order Seguno fuzzy system models each player’s emotional state which can lead to change in strategy. A set of fuzzy rules determines emotional state changes (see Table I). Each player has a subset of the rules depending on their current strategy. For instance, only Rules 1–3 are used by \( C \) players. The antecedent of each rule is based on strategy frequencies. For example, Rule 1 says a \( C \) player is “pleased” if \( x_c \) is high because many other players are contributing. Conversely, Rule 3 says a \( C \) player is “angry” if the number of \( C \) and \( P \) players are low because few are contributing. The rationale for the consequents is given in the same table.

All antecedents uses trapezoidal membership functions (see Figure 2).

![Figure 2. Membership functions used to determine emotional state. The domain of discourse is the strategy frequency.](image)

![Figure 3. Output membership functions](image)

The output membership functions are shown in Figure 3. The membership functions are singletons (constants) for a zero order Seguno fuzzy system. Only two emotional responses were considered: “pleased” and “angry”. Defuzzification can be done in various ways; we used the weighted average method. The resultant crisp value is interpreted as a emotional state change. This change is added to the player’s current emotional level. Notice “pleased” has a zero value so there is no emotional state change if a player is satisfied with the actions of others. The current strategy is changed whenever this emotional level exceeds a player’s personal emotional.

<table>
<thead>
<tr>
<th>No.</th>
<th>Player</th>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( C )</td>
<td>IF ( x_c ) is high THEN ( y ) is pleased</td>
</tr>
<tr>
<td>2</td>
<td>( C )</td>
<td>IF ( x_p ) is high THEN ( y ) is angry</td>
</tr>
<tr>
<td>3</td>
<td>( C )</td>
<td>IF ( \gamma ) is low THEN ( y ) is angry</td>
</tr>
<tr>
<td>4</td>
<td>( P )</td>
<td>IF ( x_p ) is high THEN ( y ) is pleased</td>
</tr>
<tr>
<td>5</td>
<td>( P )</td>
<td>IF ( x_d ) is high THEN ( y ) is angry</td>
</tr>
<tr>
<td>6</td>
<td>( D )</td>
<td>IF ( x_c ) is high THEN ( y ) is pleased</td>
</tr>
<tr>
<td>7</td>
<td>( D )</td>
<td>IF ( x_p ) is high THEN ( y ) is angry</td>
</tr>
<tr>
<td>8</td>
<td>( D )</td>
<td>IF ( \gamma ) is low THEN ( y ) is angry</td>
</tr>
<tr>
<td>9</td>
<td>( L )</td>
<td>IF ( x_l ) is low THEN ( y ) is pleased</td>
</tr>
<tr>
<td>10</td>
<td>( L )</td>
<td>IF ( x_l ) is high THEN ( y ) is angry</td>
</tr>
</tbody>
</table>

NOTE: \( \gamma = \frac{(x_c + x_p)}{(1 - x_i)} \) which makes \( \gamma \in (0, 1) \).

The membership functions are singletons (constants) for a zero order Seguno fuzzy system. Only two emotional responses were considered: “pleased” and “angry”. Defuzzification can be done in various ways; we used the weighted average method. The resultant crisp value is interpreted as a emotional state change. This change is added to the player’s current emotional level. Notice “pleased” has a zero value so there is no emotional state change if a player is satisfied with the actions of others. The current strategy is changed whenever this emotional level exceeds a player’s personal emotional.
level threshold. Note that all players choosing strategy $s_i$ will see the same emotional state change but that does not mean they all will change their current strategy. (Recall players do not have the same emotional threshold.) The emotional level resets back to zero if a strategy change did take place.

Once a decision is made to change to a different strategy the question then becomes what that new strategy should be. It is presumed this decision is made by a rational player which means stochastic-based choices don’t make sense. Put another way, rational players go through a deliberative process and do not make decisions—particularly those with financial implications—by coin flipping. Table II shows which choices are made by each type of player and the reasoning behind the new strategy choice.

IV. RESULTS

The simulations were run for three different distributions of player types using the parameter set shown in Table III. Figure 4 shows the results predicted by discrete replicator equations. The inset, taken from [3], shows the infinite population predictions. Clearly the finite population and infinite populations have nearly identical dynamics.

![Figure 4](image)

Figure 5 shows the results for the same player type distributions but now with the fuzzy system changing player emotional states and strategy change decisions made according to Table II reasoning. These trajectories are not nearly as smooth, but they qualitatively match the replicator predictions. Specifically, when the frequency of loners is high (black trajectory), loners initially join the PGG but eventually reverse their decision and revert back to non-participation. Even PGG players join them until the population fixates at all-$L$. If the number of defectors is very low and the frequency of loners is moderate (red trajectory), the population increases the number of contributors. (We did not record the $x_D/x_C$ ratio.) Finally, with a moderate number of all four strategies (yellow trajectory) initially $x_D$ increases but eventually, because of the decreasing number of contributors, the loner payoff becomes attractive and the population fixates at all-$L$.

One particularly interesting observation is the role punishers have in a PGG. Punishers eliminated the rock-paper-scissors oscillation of the strategies when only $D$, $C$ and $L$ players are present. Indeed, if $x_D$ is even moderately high this is sufficient to eventually fixate the population at all-$L$. Both the replicator equations and our fuzzy system method predicted this outcome.

![Figure 5](image)

V. DISCUSSION

The fuzzy rule base shown in Table I is partitioned; players use only those rules corresponding to their current strategy. All antecedents use strategy frequencies except rules 3 and 8 which use $\gamma = (x_C + x_L)/(1 - x_L)$. The reason $\gamma$ is used is these rules depend on the fraction of the population that is contributing to the public good. The consequents give the change in emotional state $\Delta E_j$, which is added to the player’s current emotional level $E_j$ for player $j$. A player makes a strategy change only when $E_j$ exceeds his personal emotional threshold which is different for each player. Making $\Delta E = 0$ if “pleased” results in no increase in $E_j$. This makes sense because there is no reason to change the current strategy if a player is satisfied with the current population strategy mixture.

Player $j$ changes strategy when $E_j$ exceeds his emotional threshold. Table II shows the factors each player type considers when deciding on the next strategy. Unlike virtually all other PGG models there is no probability associated with the new strategy choice; they are based on how a rational player might think. The thought process for each player type is modeled as follows:

- **C player**—A $C$ player checks to see if $\gamma$ is decreasing or the number of punishers is increasing. In either case contributions are decreasing. A $C$ player switches to $L$ if $\gamma \leq 1/3$ because there simply aren’t enough contributors in the current population mixture making the fixed payoff $\sigma$ more attractive. However, if $\gamma > 1/3$ then it might...
TABLE II
REACTIONS

<table>
<thead>
<tr>
<th>Player</th>
<th>Reason for Anger</th>
<th>Anger Response (see note 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>$\gamma \downarrow \text{OR } x_{P} \uparrow$</td>
<td>Switch to $L$ if $\gamma \leq \frac{1}{3}$. (see NOTE 4) Otherwise stay in PGG. Can switch to $P$ or $D$. Make choice based on the majority of these two player types.</td>
</tr>
<tr>
<td>$P$</td>
<td>$x_{0} \uparrow$</td>
<td>Leave $\beta$ fixed. Never switch to $C$ as this would be counter-productive since it tells $D$ players there is no need to change behavior. Switch to $L$ if $\gamma \leq \frac{1}{3}$. Switch to $D$ if $x_{C} = 0$ since punishment isn’t working or if $\gamma &gt; 1/3$.</td>
</tr>
<tr>
<td>$D$</td>
<td>$\gamma \downarrow \text{OR } x_{P} \uparrow$</td>
<td>$\gamma \downarrow$ means $C$ and $P$ players likely to switch in near future because returns don’t cover contributions. $x_{P} \uparrow$ means paying high punishment costs. No advantage in switching to $C$ or $P$. If $\gamma &gt; 1/3$ stay $D$. Switch to $L$ otherwise. (See NOTE 4)</td>
</tr>
<tr>
<td>$L$</td>
<td>$x_{L} \uparrow$</td>
<td>Consider joining PGG iff $\gamma &gt; 1/3$. If $x_{P}/(x_{C} + x_{L}) \geq 2$ switch to $P$ otherwise $D$. No change if $\gamma \leq 1/3$. Rationale is $L$ player decides not to join PGG after all.</td>
</tr>
</tbody>
</table>

NOTE 1: $\gamma = [(x_{C} + x_{P})/(1 - x_{L})]$
NOTE 2: There is no strategy change when agent is emotionally pleased
NOTE 3: The reason players switch to $L$ if $\gamma$ is too low is because there aren’t enough contributors to make playing the PGG worthwhile.
NOTE 4: There is no contribution for $D$ players but the rationale here is the belief $C$ and $P$ players will start switching to $L$ so $D$ players might as well do the same.

TABLE III
PARAMETER SET

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>40</td>
<td>population size</td>
</tr>
<tr>
<td>$b$</td>
<td>3</td>
<td>pot increase factor</td>
</tr>
<tr>
<td>$c$</td>
<td>1</td>
<td>contribution</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.4055</td>
<td>exponential scaling factor</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.1</td>
<td>penalty scaling factor</td>
</tr>
<tr>
<td>$\beta$</td>
<td>2</td>
<td>1st order free rider penalty</td>
</tr>
<tr>
<td>$\xi$</td>
<td>1</td>
<td>2nd order free rider penalty</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>1</td>
<td>$L$ player return</td>
</tr>
</tbody>
</table>

be more beneficial to stay in the PGG. There are two choices for a new strategy. If $P$ is in the majority then choosing to be a punisher may drive punishment levels high enough to coerce $D$ players to switch. On the other hand, if the majority are $D$ players then he would guess more punishment is likely not to have much effect. In his own self interest $D$ would be the best choice for the new strategy.

- $P$ player—A $P$ player gets upset when the fraction of $D$ players is growing. In this model the punishment level $\beta$ is fixed so increasing it is not an option. It makes no sense to switch to $C$ because that lowers the punishment seen by free riders which would be counter-productive; free riders would have even less incentive to switch in the future. If $\gamma \leq 1/3$ then there are few contributors present making the $L$ player’s fixed payoff more attractive. There are two reasons $D$ might be a better choice. If $x_{C} = 0$ there are no cooperators and, since mutation is not permitted, none will be present in the future. It is reasonable to assume punishment is not (and will not) work any longer. Conversely, if $\gamma > 1/3$, and given there are already many $D$ players, it is reasonable to assume the PGG is not going well and switching to $D$ before it is too late is the best choice.

- $D$ player—If $\gamma$ is decreasing then fewer players are contributing. Similarly, if $x_{P}$ is increasing the punishment levels are increasing. In either case defection is becoming less profitable. Switching to $C$ makes no sense if $x_{P}$ is high because punishment levels are already too high. Switching to $P$ leads to exploitation by the remaining $D$ players. The best choice is to remain defecting if $\gamma > 1/3$ since there are sufficient contributors, at the moment, to get a reasonable payoffs despite the high punishment levels and/or decreasing contributors. These circumstances would be re-evaluated the next time the emotional threshold is exceeded. If $\gamma < 1/3$ then even defection is likely to be less beneficial in the future. Under those circumstances $L$ is the best choice.

- $L$ player—If $\gamma > 1/3$ there might be a sufficient number of contributors to consider joining the PGG. In this case there are three possible new strategies to choose from. If the ratio of punishers to cooperators plus defectors is high enough then $P$ is the best choice because $C$ and $D$ players are being heavily punished. Otherwise $D$ is the best choice since switching to $C$ would lead to exploitation by the high number of defectors. However, if $\gamma \leq 1/3$ then there are too many defectors and the best choice would be to remain a loner.

In the above discussions $\gamma \leq 1/3$ appears several times. This inequality is actually $\gamma \leq c/b$. (For $c = 1$ and $b = 3$ then $\gamma \leq 1/3$). Whenever $\gamma \leq c/b$ then there aren’t enough
Figure 4 shows there is a slight difference between finite population replicator dynamics and infinite population replicator dynamics. There are no fixed points in simplex for the infinite population case except at the vertices where the population reaches fixation. In particular, when there are few $D$ players and mostly $L$ players the population reaches fixation at the $C + P$ vertex meaning the population contains only $C$ and/or $P$ players. Conversely, the finite population simplex shows with a similar initial player distribution the population evolves to a fixed point with one or more $D$ players. This fixed point arises from the constraint on Eq. (4) that the left hand side be an integer. Figure 5 shows the fuzzy system approach has an interior fixed point for that same trajectory. We can also attribute that fixed point to quantization effects.

Four strategy population trajectories reside on a 3-simplex. Due to the difficulty in drawing such a figure, one vertex was used for both $C$ and $P$ players. Nevertheless, it is possible to make some statement about the 3-simplex where each strategy has its own vertex. Clearly every vertex of a simplex is a fixed point because the population has reached fixation and mutation is not present. But there are additional fixed points as shown by the following theorem.

**Theorem.** Every point on the $x_C−x_P$ simplex boundary is a fixed point.

**Proof.** There are no $D$ or $L$ players on this boundary. Without any $D$ players a $P$ player imposes no penalties nor incurs any associated costs since there are no free riders. A $P$ or $C$ player thus has no incentive to switch strategy because their payoffs are identical.

Finally, replicator equations are widely accepted as an effective and accurate method of determining how population strategies evolve over time. Our fuzzy system method produces qualitatively similar results as those predicted by the replicator equations—in particular, the finite population replicators.

This similarity is important because it validates our fuzzy system approach. More specifically, it validates the new strategy decisions given in Table II. This is crucial for reverse engineering individual decision made during the PGG—something replicator equations cannot do. It also suggests the potential our approach might have for gaining new insight into why humans cooperate.

**VI. Future Work**

There are several directions future research efforts can take. First of all, an alternative new strategy for $P$ players might be to stay a punisher but increase $\beta$. It would be interesting to see how different punishment levels—and different costs for punishing—might affect the population mixture. Second, this preliminary work only considered two emotions: “pleased” and “angry”. Other emotional responses should be investigated, such as “annoyance”. Surprisingly, human experiments with public goods games show “guilt” plays almost no role [5], [6]. Of particular interest to this author is exploring how “trust” plays a role particularly in deciding the next strategy to use.

**REFERENCES**


